

# LOST IN TRANSLATION

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*In memory of Herb Wilf*

**ABSTRACT.** We explain the use and set grounds about applicability of algebraic transformations of arithmetic hypergeometric series for proving Ramanujan's formulae for  $1/\pi$  and their generalisations.

The principal goal of this note is to set some grounds about applicability of algebraic transformations of (arithmetic) hypergeometric series for proving Ramanujan's formulae for  $1/\pi$  and their numerous generalisations. The technique was successfully used in quite different situations [7, 16, 18, 19, 20] and was dubbed as 'translation method' by J. Guillera, although the name does not give any clue about the method itself.

Consider the following problem: *Show that*

$$\sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} (3 + 40n) \cdot \frac{1}{28^{4n}} = \frac{49}{3\sqrt{3}\pi}. \quad (1)$$

*Step 0.* It comes as a useful rule: prior to any attempts to prove an identity verify it numerically. The convergence of the series on the left-hand side of (1) is reasonably fast (more than 3 decimal places per term), so you shortly convince yourself that the both sides are

$$3.001679541740867825117222046370611403163548615329487998574326 \dots$$

*Step 1.* Series of the type given in (1) should be quite special. With a little search you identify

$$\sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \left( \frac{x}{256} \right)^n = {}_3F_2 \left( \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \middle| x \right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n (1)_n} \frac{x^n}{n!}, \quad (2)$$

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*Date:* 26 September 2012.

*2010 Mathematics Subject Classification.* Primary 33C20; Secondary 11F03, 11F11, 11Y60, 33C45.

*Key words and phrases.*  $\pi$ , Ramanujan, arithmetic hypergeometric series, algebraic transformation, modular function.

The author is supported by the Australian Research Council.

a hypergeometric series, where the notation  $(a)_n$  (*Pochhammer's symbol* or *shifted factorial*) stands for  $\Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ . A generalised hypergeometric series

$${}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{x^n}{n!}$$

is an object of intensive study since Euler [2, 17]; one of its important properties is the linear differential equation

$$\left(\left(x \frac{d}{dx}\right) \prod_{j=2}^m \left(x \frac{d}{dx} + b_j - 1\right) - x \prod_{j=1}^m \left(x \frac{d}{dx} + a_j\right)\right) F = 0 \quad (3)$$

satisfied by the series. The required identity (1) can be therefore transformed to the more conceptual form

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{3+40n}{7^{4n}} = \left(3 + 40x \frac{d}{dx}\right) {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix} \middle| x\right) \Big|_{x=1/7^4} = \frac{49}{3\sqrt{3}\pi}. \quad (4)$$

*Step 2.* Convince yourself that identities of the wanted type are known in the literature. In fact, they are known for almost a century after Ramanujan's publication [15]; identity (1) is equation (42) there. Ramanujan did not indicate how he arrived at his series but left some hints that these series belong to what is now known as 'the theories of elliptic functions to alternative bases'. The first proofs of Ramanujan's identities and their generalisations were given by the Borweins [5] and Chudnovskys [8]. Those proofs are however too lengthy to be included here. Note that Ramanujan's list in [15] does not include the slowly convergent example

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (1+4n) (-1)^n = \left(1 + 4x \frac{d}{dx}\right) {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| x\right) \Big|_{x=-1} = \frac{2}{\pi}, \quad (5)$$

which was shown to be true by G. Bauer [3] already in 1859. Bauer's proof makes no reference to sophisticated theories and is much shorter, although does not seem to be generalisable to the other entries from [15]. In fact, D. Zeilberger assisted by his automatic collaborator S. B. Ekhad [9] came up in 1994 with a short proof of (5) verifiable by a computer. The key is a use of a simple telescoping argument (this part is completely automated by the great Wilf–Zeilberger (WZ) machinery [14]) and an advanced theorem due to Carlson [2, Chap. V]; the proof is reproduced in [21]. Quite recently, J. Guillera advocated [10, 11, 12, 13] the method from [9] and significantly extended the outcomes; he showed, for example, that many other Ramanujan's identities for  $1/\pi$  can be proven completely automatically. Note however that (1) is one of 'WZ resistant' identities. To overcome this technical difficulty, below we reduce the identity to the simpler one (5).

*Step 3.* Use your favourite computer algebra system (CAS) to verify the hypergeometric identity

$${}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| x\right) = r \cdot {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix} \middle| y\right) \quad (6)$$

where  $y = y(x) = -\frac{1}{1024}x^3 + O(x^4)$  and  $r = r(x) = 1 + \frac{1}{8}x + \frac{27}{512}x^2 + O(x^3)$  are algebraic functions determined by the equations

$$\begin{aligned} & (x^2 - 194x + 1)^4 y^4 \\ & + 16(4833x^6 + 2029050x^5 + 47902255x^4 - 92794388x^3 \\ & + 47902255x^2 + 2029050x + 4833)xy^3 \\ & - 96(3328x^6 - 623745x^5 + 3837060x^4 - 6470150x^3 \\ & + 3837060x^2 - 623745x + 3328)xy^2 \\ & + 256(1024x^6 - 1152x^5 + 225x^4 - 2x^3 + 225x^2 - 1152x + 1024)xy + 256x^4 = 0 \end{aligned}$$

and

$$\begin{aligned} & (x^2 - 194x + 1)^2 r^8 + 4(61x^2 + 25798x + 61)(x - 1)r^6 \\ & + 486(41x^2 - 658x + 41)r^4 + 551124(x - 1)r^2 + 531441 = 0. \end{aligned}$$

To do this you (and your CAS) are expected to use the linear differential equations (3) for the involved hypergeometric functions and generate any-order derivatives of  $y$  and  $r$  with respect to  $x$  by appealing to the implicit functional equations. To summarise, you have to check that the both sides of (6) satisfy the same (3rd order) linear differential equation in  $x$  with algebraic function coefficients and then compare the first few coefficients in the expansions in powers of  $x$ . Note that  $x = -1$  corresponds to  $y = 1/7^4$  (cf. (5) vs. (4)), and this is the reason behind considering the sophisticated functional identity (6).

The task on this step does not look humanly pleasant, and there is a (casual) trick to verify (6) by parameterising  $x$ ,  $y$  and  $r$ :

$$\begin{aligned} x &= -\frac{4p(1-p)(1+p)^3(2-p)^3}{(1-2p)^6}, & y &= \frac{16p^3(1-p)^3(1+p)(2-p)(1-2p)^2}{(1-2p+4p^3-2p^4)^4}, \\ r &= \frac{(1-2p)^3}{1-2p+4p^3-2p^4}. \end{aligned}$$

Choosing  $p = (1 - \sqrt{45 - 18\sqrt{6}})/2$  we obtain  $x = -1$  and  $y = 1/7^4$ . (The modular reasons behind this parametrisation can be found in [4, Lemma 5.5 on p. 111] where the  $p$  there is the negative of our  $p$ .)

*Step 4.* By differentiating identity (6) with respect to  $x$  and combining the result with (6) itself we see that

$$\left(a + bx \frac{d}{dx}\right) {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| x\right) = \left(a + bx \frac{dr}{dx} + b \frac{rx}{y} \frac{dy}{dx} \cdot y \frac{d}{dy}\right) \cdot {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix} \middle| y\right); \quad (7)$$

again, the derivatives  $dy/dx$  and  $dr/dx$  are read from the implicit functional equations. An alternative (but simpler) way is using the parametrisations  $x(p)$ ,  $y(p)$  and  $r(p)$ . Taking  $a = 1$ ,  $b = 4$  and  $x = -1$  in (7) you recognise the familiar Bauer's (WZ easy) identity (5) on the left-hand side; the right-hand side counterpart is nothing but (4).

*Comments.* The story exposed above is general enough to be used in other situations for proving *some* other formulae for  $1/\pi$ . The setup can be as follows. Assume we already have an identity

$$\left(a + bx \frac{d}{dx}\right) F(x) \Big|_{x=x_0} = \mu,$$

where  $a, b, x_0$  and  $\mu$  are certain (simple or at least arithmetically significant) numbers, and  $F(x)$  is an (arithmetic) series. Furthermore, assume we have a transformation  $F(x) = rG(y)$  with  $r = r(x)$  and  $y = y(x)$  differentiable at  $x = x_0$ . Then

$$\left(\hat{a} + \hat{b}y \frac{d}{dy}\right) G(y) \Big|_{y=y_0} = \mu,$$

where

$$\hat{a} = a + bx \frac{dr}{dx} \Big|_{x=x_0}, \quad \hat{b} = b \frac{rx}{y} \frac{dy}{dx} \Big|_{x=x_0}, \quad \text{and} \quad y = y_0.$$

There is, of course, no magic in this result: it is just the standard ‘chain rule’.

The applicability of this simple argument heavily rests on existence of transformations like (6). This in turn is based on the modular origin [5, 6, 8, 21] of Ramanujan’s identities for  $1/\pi$ : any such identity can be written in the form

$$\left(a + bx \frac{d}{dx}\right) F(x) \Big|_{x=x_0} = \frac{c}{\pi}, \quad a, b, c, x_0 \in \mathbb{Q}, \quad (8)$$

where  $F(x)$  is an *arithmetic hypergeometric series* [23] satisfying a 3rd order linear differential equation. In other words, for a certain modular function  $x = x(\tau)$  (not uniquely defined!) the function  $F(x(\tau))$  is a modular form of weight 2. The theory of modular forms provides us with the knowledge that any two modular forms are algebraically dependent; thus, whenever we have another arithmetic hypergeometric series  $G(y)$  and a related modular parametrisation  $y = y(\tau)$ , the modular functions  $y(\tau)$  and  $G(y(\tau))/F(x(\tau))$  are algebraic over  $\mathbb{Q}[x(\tau)]$ . Another warrants of the theory is an algebraic dependence over  $\mathbb{Q}$  of  $x(\tau)$  and  $x((A\tau + B)/(C\tau + D))$  for any  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Q})$ . On the other hand, there is no other source known for such algebraic dependency; the functions  $x(\tau)$  and  $x(A\tau)$ ,  $A > 0$ , are algebraically dependent if and only if  $A$  is rational.

The above arithmetic constraints impose the natural restriction on  $\tau_0$  from the upper half-plane  $\text{Re } \tau > 0$  to satisfy  $x(\tau_0) = x_0$  in (8). Namely,  $\tau_0$  is an (imaginary) quadratic irrationality,  $\tau_0 \in \mathbb{Q}[\sqrt{-d}]$  for some positive integer  $d$ . But then  $(A\tau_0 + B)/(C\tau_0 + D)$  belongs to the same quadratic extension of  $\mathbb{Q}$  for any  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Q})$ , so whatever transformation  $F(x) = rG(y)$  (of modular origin) we use, the modular arguments of  $x(\tau)$  and  $y(\tau)$  have to be tied by an  $SL_2(\mathbb{Q})$  linear-fractional transform. In the examples (4) and (5) we have both arguments belonging to  $\mathbb{Q}[\sqrt{-2}]$ , therefore an algebraic transformation must exist, and this is confirmed by (6) mapping the corresponding  $x(\tau_0) = -1$  into  $y(3\tau_0) = 1/7^4$  where  $\tau_0 = (1 + \sqrt{-2})/2$ . There is

however no way known to ‘translate’ identities (4) and (5) to either

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (1+6n) \frac{1}{4^n} = \frac{4}{\pi}$$

or

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} (13591409 + 545140134n) \cdot \frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\sqrt{10005}\pi},$$

as the corresponding modular arguments lie in the fields  $\mathbb{Q}[\sqrt{-3}]$  and  $\mathbb{Q}[\sqrt{-163}]$ , respectively. We refer the interested reader to [6] for exhausting lists of ‘rational’ (in the sense of  $x_0$ ) identities which express  $1/\pi$  by means of general hypergeometric-type series; the details of the modular machinery are greatly explained there.

In a sense, to make the ‘translation method’ work we first should carefully examine the underlying modular parametrisations. On the other hand, there are situations when we know (or can produce [1]) the algebraic transformations without having modularity at all. These are particularly useful in the context of similar formulae for  $1/\pi^2$  recently discovered by Guillera [10, 11, 13].

There is a  $p$ -adic counterpart of the Ramanujan-type identities for  $1/\pi$  and  $1/\pi^2$  which we review in [22]. It seems likely that the algebraic transformation machinery is generalisable to those situations as well but, for the moment, no single example of this is known.

**Acknowledgements.** I would like to thank Shaun Cooper for his useful suggestions which helped me to improve on an earlier draft of this note.

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